Anatomy of Singularities in Geometrical and Physical Systems

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Abstract

This paper explores the geometrical and physical principles underlying singularities in expanding systems, anisotropic spaces, and gravitational phenomena. Starting with the axiomatic definition of coordinates in infinite lines and expanding spaces, the study delves into the mathematical representation of uniform and non-uniform expansions, culminating in a detailed discussion of conical and spherical geometries. Furthermore, the paper examines the concept of hollow singularities in the context of planetary gravitation, harmonic motion, and relativistic constraints near event horizons. By integrating classical mechanics, relativity, and cosmological concepts, this work provides a comprehensive framework to understand the dynamics of singularities and their manifestations in different scales of the universe.

Keywords: singularities, coordinate systems, anisotropy, gravitational dynamics, Schwarzschild metric, cosmology, harmonic motion.

1 Introduction

Singularities represent critical points in mathematical and physical systems where conventional laws or definitions cease to apply or become undefined. Their study has profound implications across various fields, including geometry, cosmology, and general relativity. From the uniform expansion of lines to the anisotropic behavior of non-homogeneous spaces, singularities serve as both a mathematical challenge and a gateway to deeper physical insights.

This paper begins by addressing the axioms of coordinates and their application to infinite and expanding systems. Using geometrical representations, such as cones and spheres, it illustrates how spatial compression or expansion can be perceived by observers under different conditions. The discussion transitions to gravitational dynamics, with particular emphasis on the behavior of objects within spherical planets and near black holes. Finally, the concept of hollow singularities is introduced, offering a novel perspective on inaccessible regions within gravitational systems.

2 1. The Concept of Singularities

Singularities are regions of space-time where gravitational forces become so intense that the curvature of space-time becomes infinite. These regions are often hidden from view, as they are located within black holes, whose event horizons prevent any information from escaping.

In general relativity, a singularity occurs when the curvature of space-time, described by the Riemann curvature tensor, reaches infinite values. This is typically the case at the center of black holes, where matter is compressed to a point of infinite density. The most well-known example of a singularity is the one predicted to exist at the center of a black hole.

The mathematical representation of a singularity can be described by the Schwarzschild solution to Einstein's field equations, which is the solution for a non-rotating, uncharged black hole:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Where:

- ds^2 represents the space-time interval,
- M is the mass of the central object,
- r is the radial distance from the center,
- G is the gravitational constant,
- t, θ, ϕ are the time and angular coordinates.

The singularity occurs at $r = 0$, where the space-time curvature becomes infinite, and thus the equations cease to provide meaningful physical predictions.

3 Challenges in Studying Singularities

Despite significant progress in understanding black holes and singularities, much remains unknown. The most significant challenge in singularity research is that the current laws of physics, including general relativity, break down at these extreme points. Additionally, the singularity itself is hidden within the event horizon, making it impossible to observe directly.

4 2. Event Horizons and Gravitational Redshift

An event horizon is the boundary surrounding a black hole, beyond which nothing, not even light, can escape the gravitational pull. The concept of the event horizon is crucial for understanding the behavior of objects near a black hole, as it defines the region from which no information can return to the outside universe.

When an object approaches the event horizon, it undergoes significant changes in its perception of time and space. This is due to the effects of gravitational time dilation, which causes time to pass more slowly near massive objects. As the object gets closer to the event horizon, the dilation becomes more extreme, with time appearing to freeze at the event horizon from the perspective of a distant observer.

The gravitational redshift, a phenomenon predicted by general relativity, describes the stretching of light's wavelength as it escapes from a massive object. This effect is particularly pronounced near the event horizon, where the wavelength of light increases significantly. The mathematical expression for gravitational redshift is given by:

$$z = \frac{\lambda_{\text{observed}} - \lambda_{\text{emitted}}}{\lambda_{\text{emitted}}} = \sqrt{1 - \frac{2GM}{rc^2}}$$

Where:

- z is the redshift,
- $\lambda_{\text{observed}}$ and λ_{emitted} are the observed and emitted wavelengths of light,
- r is the radial distance from the black hole's center,
- G is the gravitational constant,
- M is the mass of the black hole,
- c is the speed of light.

5 3. Gravitational Collapse and Black Hole Formation

Gravitational collapse occurs when a massive star exhausts its nuclear fuel and no longer produces enough pressure to counteract its own gravity. The star then contracts under its own weight, and if the mass is large enough, it will collapse into a singularity, forming a black hole.

The mathematical description of gravitational collapse is often modeled by the Tolman-Oppenheimer-Volkoff (TOV) equation, which governs the equilibrium of a spherically symmetric object under its own gravity:

$$\frac{dP}{dr} = -\frac{G(\rho + P)(m + 4\pi r^3 P)}{r(r - 2GM)}$$

Where:

- P is the pressure,
- ρ is the density,
- r is the radial distance from the center of the object,
- m is the mass within radius r,
- G is the gravitational constant.

The equation describes how pressure and density change within the collapsing star as it becomes increasingly compact. When the density becomes infinitely large, the collapse forms a singularity, and the star becomes a black hole.

6 Event Horizon and the Schwarzschild Radius

The radius of the event horizon is known as the Schwarzschild radius, and it defines the boundary beyond which no object or light can escape the gravitational pull of the black hole. The Schwarzschild radius is given by:

$$r_s = \frac{2GM}{c^2}$$

Where:

- r_s is the Schwarzschild radius,
- M is the mass of the black hole,
- G is the gravitational constant,
- c is the speed of light.

7 4. Hollow Singularities and the Concept of Gravitational Opacity

The concept of hollow singularities arises from the idea that certain black holes may have event horizons that create regions of space-time where the interior is inaccessible to any trajectories of matter or light. This idea leads to what is called gravitational opacity—an invisible barrier that prevents any information from reaching the outside universe.

In some theoretical scenarios, a massive object approaching the event horizon could experience time dilation to such an extent that it appears to freeze at the horizon from the perspective of an external observer. However, from the object's own perspective, it continues to fall into the black hole.

$$\frac{dt}{d\tau} = \sqrt{1 - \frac{2GM}{rc^2}}$$

Where:

- τ is the proper time of the falling object,

- r is the radial distance from the black hole's center.

This equation illustrates how the passage of time slows for an object as it approaches the event horizon. At the event horizon, time appears to stop from the viewpoint of an external observer.

8 Axioms on Coordinates

8.1 Definition of an Infinite Line with Uniform Expansion

Consider an infinite line whose length changes over time in a uniform manner. The distance $s(t)$ between two points A and B can be modeled as:

$$s(t) = s_0 + v \cdot t,$$

where:

- $s(t)$ is the length of the line at time t,
- s_0 is the initial length of the line at time $t = 0$,
- v is the rate of expansion (constant), which describes the uniform growth,
- t is the time.

This model reflects a homogeneous expansion where both the unit of measure and the proportion between the segments remain constant. This is equivalent to the idea of scale invariance in homogeneous systems, as discussed by Weyl (1923).

8.2 Non-Homogeneous Expansion

When the expansion of the line varies at different positions, the rate of growth v becomes position-dependent. The distance between two points can be described by the integration of $v(x)$:

$$s(x, t) = \int_{x_1}^{x_2} v(x)\, dx,$$

where:

- x_1 and x_2 are the initial and final positions on the line,
- $v(x)$ is the rate of expansion as a function of x.

If $v(x) = k \cdot x$, where k is a constant, we have:

$$s(x,t) = \int_{x_1}^{x_2} k \cdot x \, dx = \frac{k}{2} \left(x_2^2 - x_1^2\right).$$

In this case, the expansion is faster at positions farther from the origin $x = 0$, creating a perception of spatial anisotropy.

8.3 Geometry of the Cone

A vector $\vec{V}$ pointing towards a point P generates a geometric configuration resembling a cone. The distance along a radius r can be described as:

$$l(r) = \int_0^r f(r') \, dr',$$

where:

- $l(r)$ is the length measured along the radius up to r,
- $f(r')$ is a function describing the rate of expansion/compression along the radius r'.

If $f(r') = \frac{1}{r'}$, the distance becomes:

$$l(r) = \int_0^r \frac{1}{r'} \, dr' = \ln(r).$$

This means that the space near the point P (when $r \to 0$) is compressed on a logarithmic scale.

To observe the perceived expansion or contraction along the cone, we compare the ratios between the distances:

$$\text{Scale Ratio} = \frac{l_{\text{base}}}{l_P} = \frac{\ln(R_{\text{base}})}{\ln(R_P)}.$$

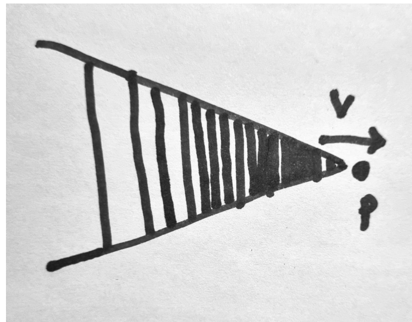

Figure 1: An image showing the cone conception of space.

8.4 Spherical Geometry and Compression

When replacing the cone with a sphere, the radial distance within the sphere can be described as:

$$l(r) = \int_0^r \frac{1}{r'}\, dr' = \ln(r).$$

The compression perceived by an external observer near the edge (when $r \to R$) reflects the same logarithmic progression but at a more pronounced scale.

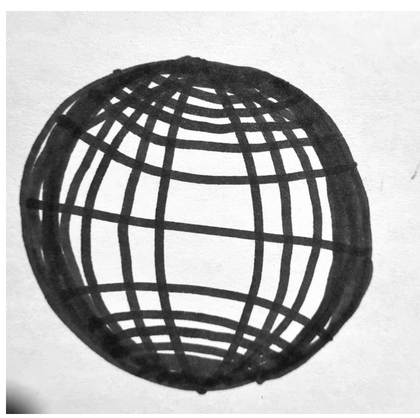

Figure 2: An image showing the conception of singularity spherics horizon.

If the time required to cross a unit of measure Δr is constant (Δt), the velocity $v(r)$ of an internal vector becomes:

$$v(r) = \frac{\Delta r}{\Delta t} \propto \frac{1}{r}.$$

This creates a deceleration effect noticeable as $r \to 0$, reflecting convergent series behavior:

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

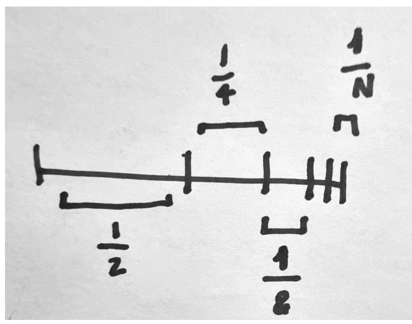

Figure 3: An image showing the increase of units of measuring in the space.

9 Occasional Singularities

9.1 Gravity on a Spherical Planet with Uniform Density

On a spherical planet with uniform density, the gravitational force $F(r)$ inside is proportional to the distance r from the center:

$$F(r) = -\frac{GM(r)}{r^2},$$

where:

- G is the gravitational constant,
- $M(r)$ is the mass contained within radius r, which, for uniform density ρ, is $M(r) = \frac{4}{3}\pi r^3 \rho$.

Substituting $M(r)$, we get:

$$F(r) = -\frac{G \cdot \frac{4}{3}\pi r^3 \rho}{r^2} = -\frac{4}{3}\pi G \rho r.$$

This indicates that the gravitational force increases linearly with r.

9.2 Simple Harmonic Motion Inside the Planet

An object falling inside a tunnel connecting two opposite points on the planet will experience simple harmonic motion. The acceleration $a(r)$ is given by:

$$a(r) = \frac{F(r)}{m} = -\frac{4}{3}\pi G \rho r,$$

where m is the mass of the object. The equation of motion is:

$$\ddot{r} + \omega^2 r = 0,$$

with angular frequency:

$$\omega = \sqrt{\frac{4}{3}\pi G \rho}.$$

The solution to this equation is:

$$r(t) = R\cos(\omega t),$$

where R is the maximum amplitude (the initial distance from the surface to the core).

9.3 Relativity and Maximum Velocity

If the planet has a sufficiently large radius, the velocity of the falling object may approach the speed of light c. In this case, the velocity is limited by:

$$v(r) = \frac{c}{\sqrt{1 + \frac{r^2}{R^2}}}.$$

where R is the maximum radius of the planet.

9.4 Event Horizon and Singularity

For black holes, the Schwarzschild metric describes the behavior of proper time τ as a function of coordinate time t:

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{r_s}{r}},$$

where r_s is the Schwarzschild radius. When $r \to r_s$, $d\tau \to 0$, creating an impenetrable barrier for external observers.

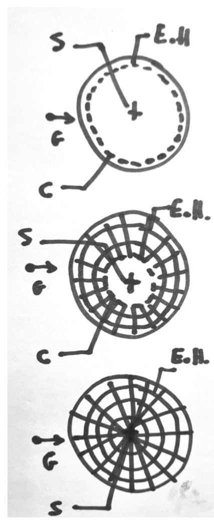

Figure 4: An image showing the donut's event horizon. Here, G represents the gravitational force, C denotes the speed of light when the body accelerates, E.H. stands for event horizon, S refers to the singularity, and the "+" symbol marks the center of the Event Horizon.

10 Conclusion

This article explores the complex relationship between space expansion, gravitational forces, and the behavior of singularities in a variety of scenarios. The primary focus is on the mathematical models that describe the dynamic behavior of expanding lines and spherical geometries, particularly in relation to the behavior of gravity in a spherical planet and the impact of non-homogeneous space expansion. The model of a uniform expansion of space

and its more complex, position-dependent counterpart offers a framework to understand how distances and geometries change over time. The inclusion of the concept of black holes and the behavior of objects near singularities adds a critical layer to the discussion, emphasizing the limits imposed by general relativity and the curvature of spacetime.

In the next image, the red line serves as a visual representation of the boundary beyond which access to the inner regions of the event horizon becomes impossible. This boundary marks the transition from accessible space to the regions that are effectively isolated from the external universe. The yellow area highlights these inaccessible regions, emphasizing the profound influence that the singularity exerts on the surrounding spacetime. It is crucial to note that the center of the singularity does not reside in a singular, well-defined point but instead exists as an infinitesimal concentration of spacetime lines. This suggests that the singularity may not be a localized entity, but rather an extreme distortion that affects the surrounding space in ways that challenge our traditional understanding of point-based singularities. The concept of singularities, and particularly their geometry as depicted in this image, pushes the boundaries of current physical theory, offering new perspectives on how the fabric of spacetime behaves under extreme conditions. These insights pave the way for further exploration into the nature of black holes and the mysteries of spacetime in the presence of singularities

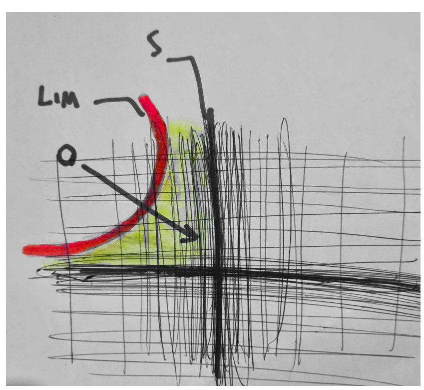

Figure 5: Conception of spacetime metrics around singularities, making regions inaccessible and timeless.

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